Regular Representations: graphs, digraphs, oriented graphs, and coloured graphs

Joy Morris
joint work with Pablo Spiga and others

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Answer [Frucht, 1938]

Yes; in fact, there are infinitely many such graphs for any group G.

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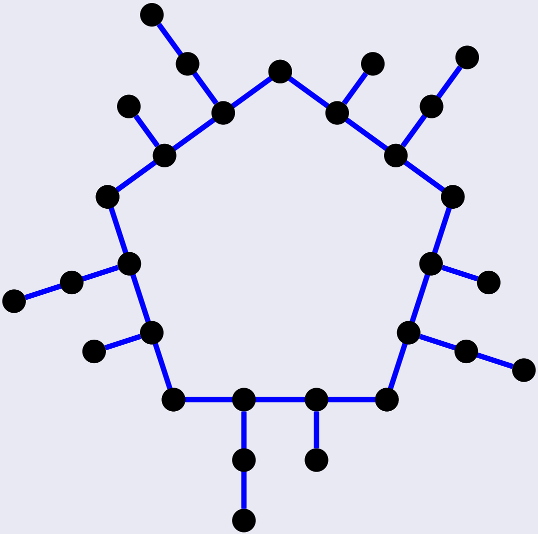
Example: $\mathbb{Z}_5$



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Cayley digraphs

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Notice that left-multiplying by h preserves adjacency, so the regular representation of G is in $\mathrm{Aut}(\Gamma)$.

Cayley graph examples

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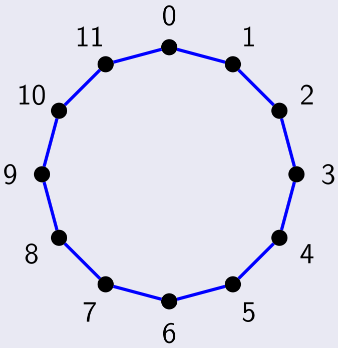

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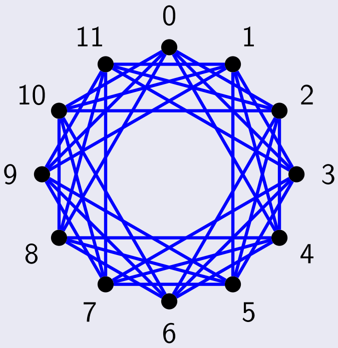

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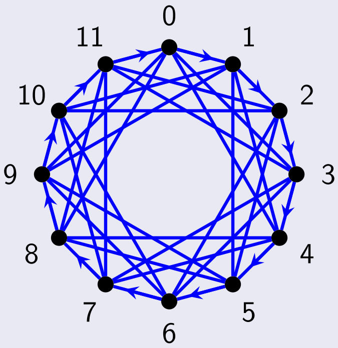

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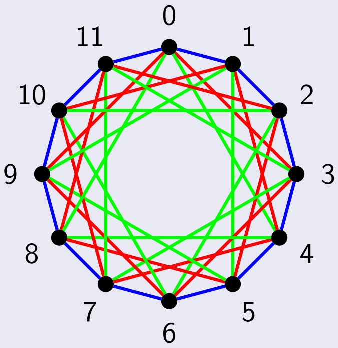

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In particular, since S is inverse-closed, if there is an automorphism of G that takes every element of G to itself or its inverse, then Γ has this extra automorphism.

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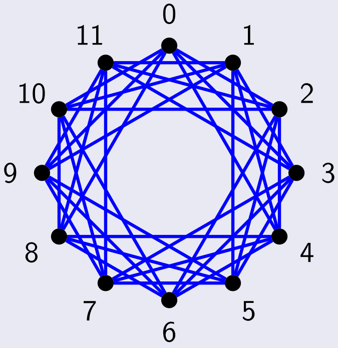

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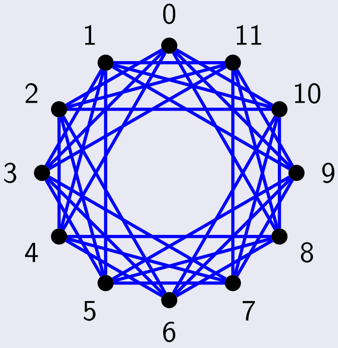

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Theorem (Hetzel 1976, Godsil 1981)

With the exception of these two infinite families and 13 other groups of order at most 32, every group has a GRR.

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So Babai's question is, what groups admit an ORR? As in the case of GRRs, there is an obstruction.

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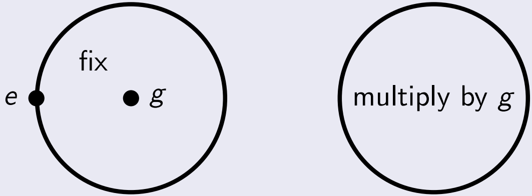
Extra graph automorphism:



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Observation (Babai, 1980)

Generalised dihedral groups do not admit ORRs.

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- *a 2-group, with additional conditions.*

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- one of 11 exceptions of order at most 64.

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Proof.

Since the graph is vertex-transitive, the hypothesis implies that any time any one vertex is fixed, all of its neighbours via elements of X are fixed.

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Since the graph is vertex-transitive, the hypothesis implies that any time any one vertex is fixed, all of its neighbours via elements of X are fixed. Since $\langle X \rangle = G$ the graph is connected, so fixing one vertex forces every vertex to be fixed.

Major tool

Lemma (Nowitz-Watkins, 1972)

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A GRR for C_2^n where $n \geq 6$ (Imrich, 1970)

Let $C_2^n = \langle a_1, \ldots, a_n \rangle$.
Let $S = \{a_1, \ldots, a_n, a_1 a_2, \ldots a_{n-1} a_n, a_1 a_2 a_{n-2} a_{n-1}, a_1 a_2 a_{n-1} a_n\}$.
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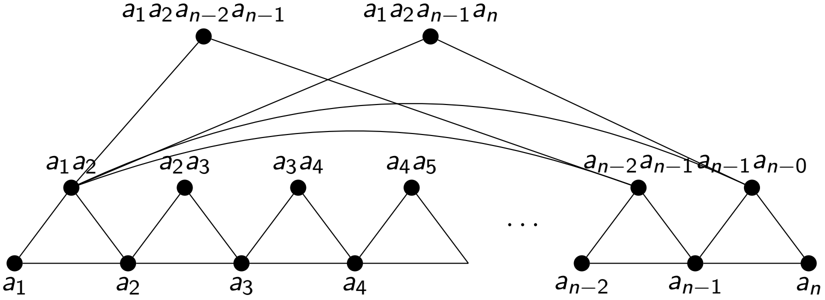

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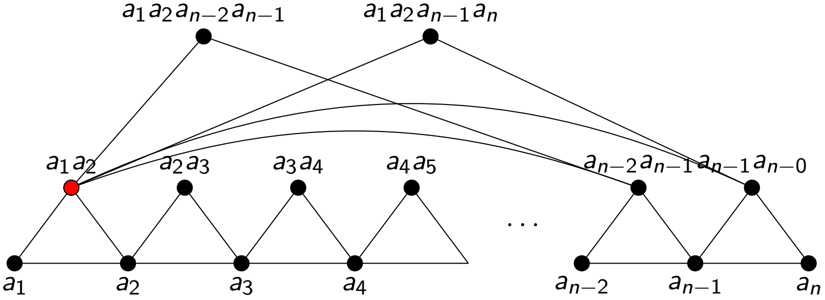

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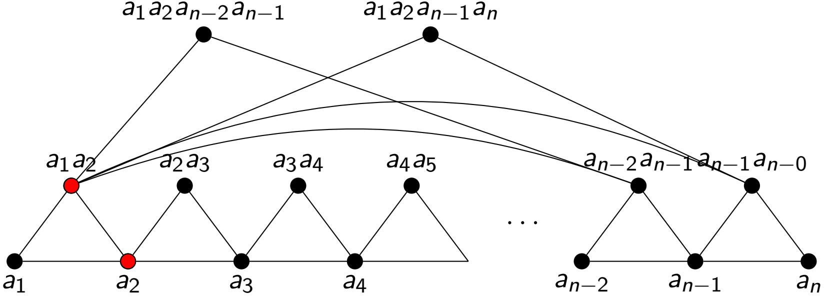

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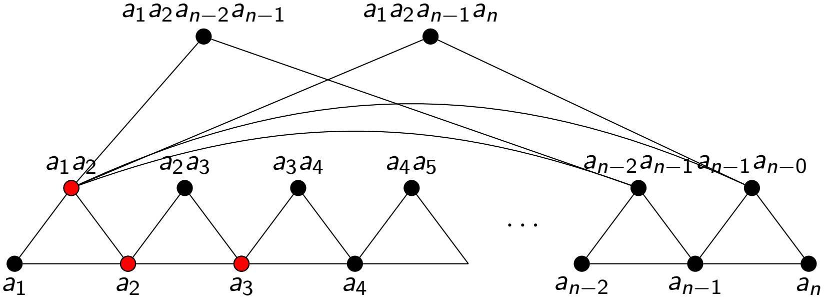

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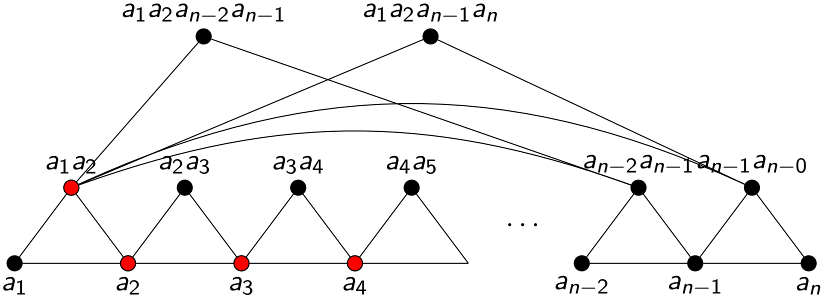

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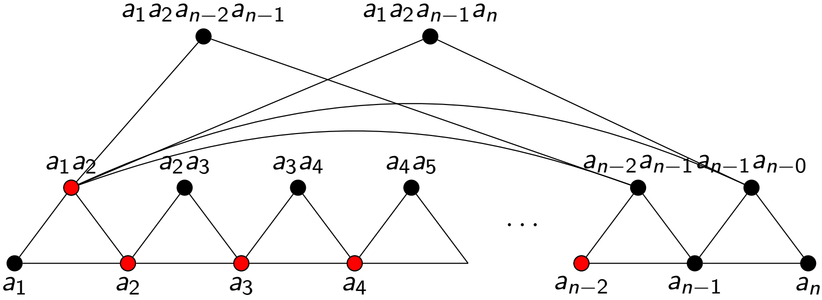

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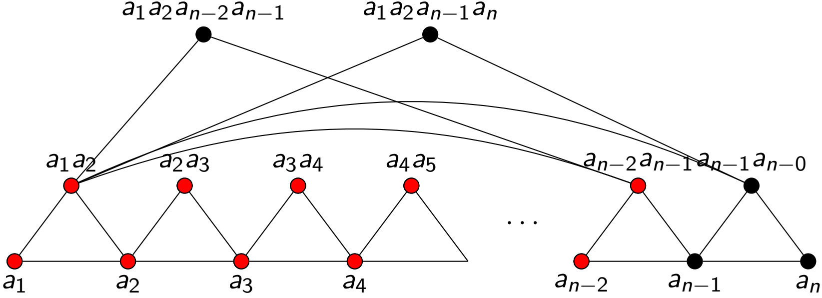

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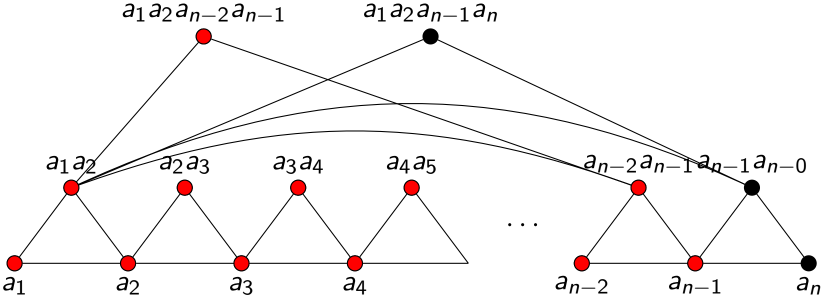

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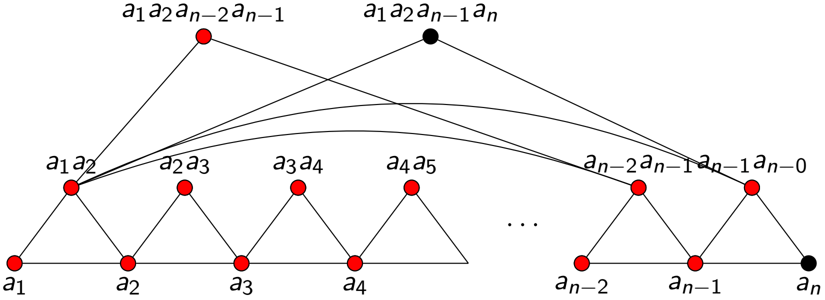

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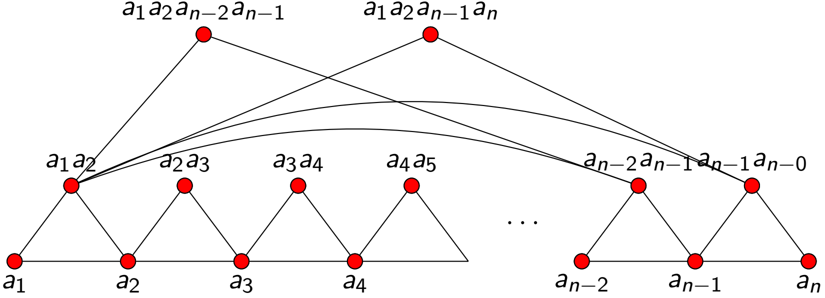

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Main ideas of proof

We first consider all groups that can be generated by at most 3 elements, and either use a small generating set to construct an ORR, or show that the group does not admit an ORR.

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Using CFSG and induction on the smallest size of a generating set, we show that every non-solvable group admits such a generating set.

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- the product of any two elements of Vg lies in V, so has order 2.

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- Observe that this implies T is fixed setwise, so every element of T is fixed pointwise, and therefore by Nowitz-Watkins, the Cayley graph is an ORR.

Edge colouring

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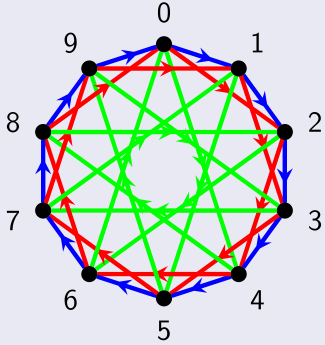

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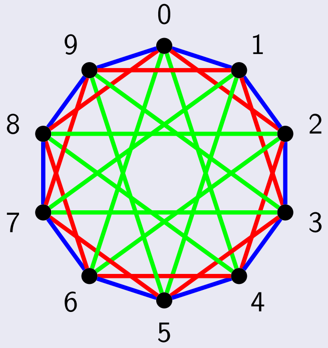

$n = 10$, $S = \{1, 9, 2, 8, 4, 6\}$.

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A more interesting question, therefore, relates to the number of coloured GRRs for any given group.

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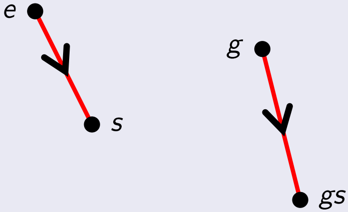
Suppose that the arc from g to gs is coloured red, so every s-arc is red. This is the only red arc from g, so the preservation of colours forces $\alpha(gs) = \alpha(g)s = gs$.



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However, the proof will work if every element of S is an involution.

Also...

The condition of connectedness is necessary.

Coloured GRRs

If we also allow graph automorphisms that come from group automorphisms but preserve edge colours, I have been studying this question with Ted Dobson, Brandon Fuller, Ademir Hujdurović, Klavdija Kutnar, Luke Morgan, Dave Morris, and Gabriel Verret (in various combinations), calling it the CCA (Cayley Colour Automorphism) problem.

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- I have no idea what is known for infinite groups and graphs.

Thank you!