The intersection property on k-orbit polytopes

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Maniplex

Definition

Let $\mathcal{M}$ be connected simple graph with a proper edge coloring $c : M \to \{0, \ldots, n-1\}$. We say that $\mathcal{M}$ is an n-maniplex if whenever $|i-j| > 1$, all paths of length 4 that alternate colors between i and j are closed.

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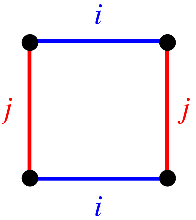

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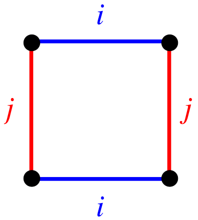

Examples of maniplexes are the flag graphs of polytopes and maps, hence the vertices of a maniplex are often refered to as flags.

Polytopality of maniplexes

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A maniplex satisfies the path intersection property (PIP) if whenever two vertices Φ and Ψ are connected by some path with colors in a set I and another path with colors in J, then they are connected by a path with colors in $I \cap J$.

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Theorem

Garza-Vargas and Hubard, 2018 *A maniplex $\mathcal{M}$ is the flag graph of a polytope if and only if it satisfies the PIP.*

Symmetry type graph

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Let $\mathcal{M}$ be a maniplex and let $\Gamma(\mathcal{M})$ be its automorphism group. Let $H < \Gamma(\mathcal{M})$. We define the symmetry type graph (STG) of $\mathcal{M}$ with respect to H to be the quotient $\mathcal{T}(\mathcal{M}, H) := \mathcal{M}/H$.

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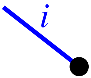
If there's an edge of color i between Φ and Ψ on $\Rightarrow$ there's an edge of color i between their orbits. If they are on the same orbit we draw a semi-edge on that orbit (not a loop).



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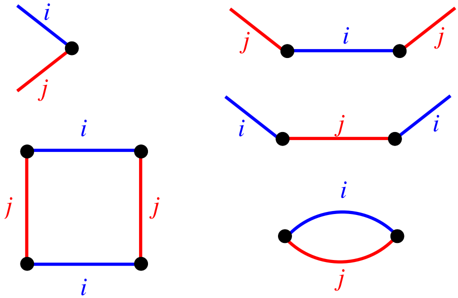
Connected components of the graph induced by edges of colors colors i and j are one of the following:



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Given a vertex x in X, we denote the (fundamental) group of closed paths based at x as $\Pi^x(X)$.

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Let X be a graph. A voltage assignment is a (groupoid) anti-homomorphism $\xi : \Pi(X) \to G$ where G is a group. In this case, G is called the voltage group.

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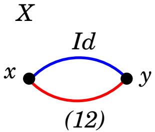
We only need to define the voltage of the arcs of X to define the voltage of all its paths. The voltage of a path is the product of the voltages of its arcs in reverse order.

Voltages

Given a graph X with a voltage assignment ξ, we may construct a "bigger" derived graph X^ξ in the following way:



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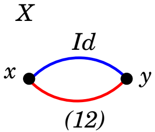

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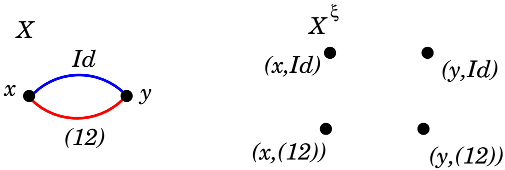

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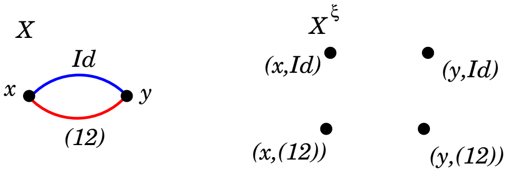

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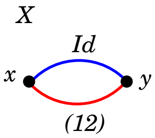

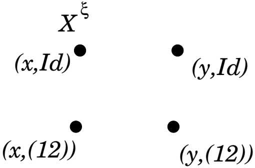

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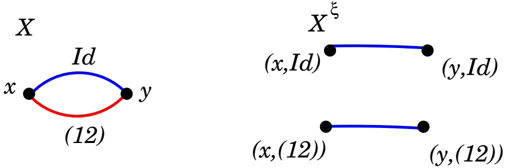

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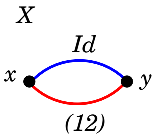

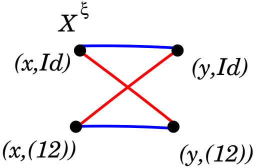

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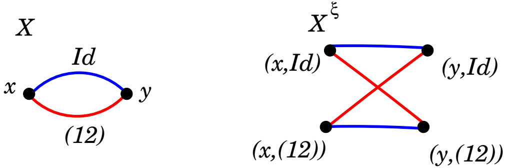

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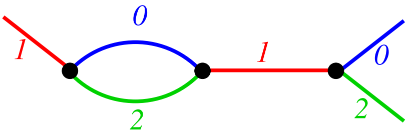

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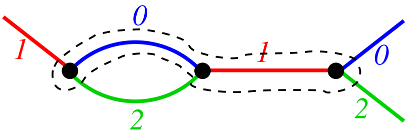

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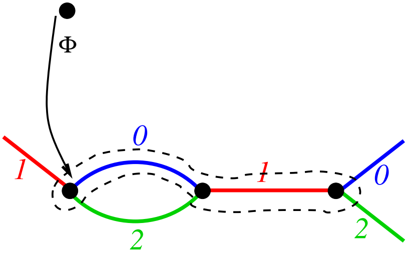

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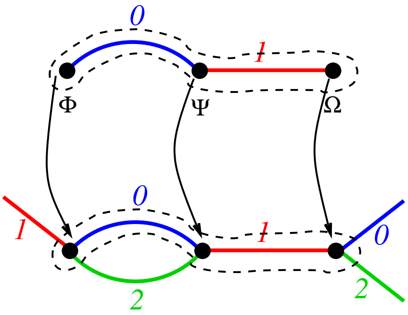

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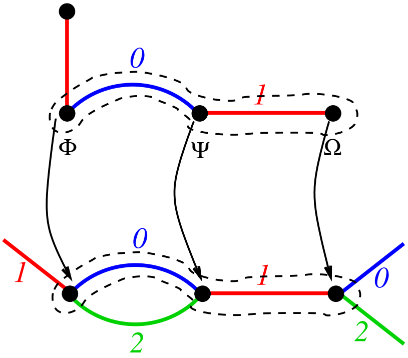

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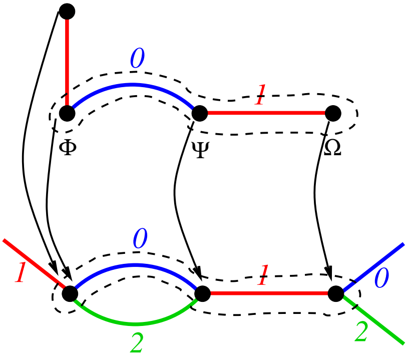

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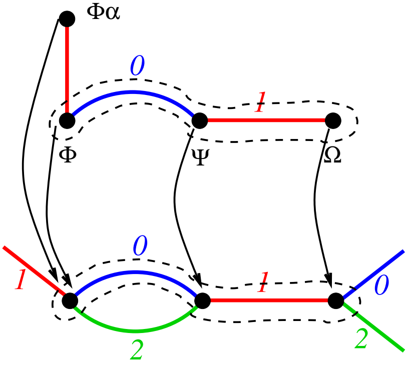

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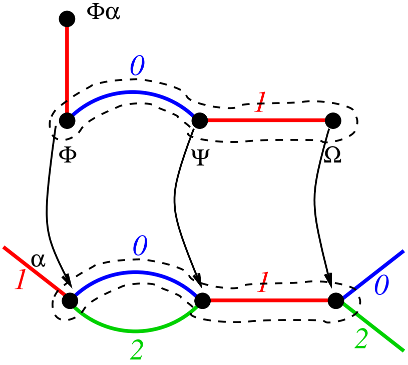

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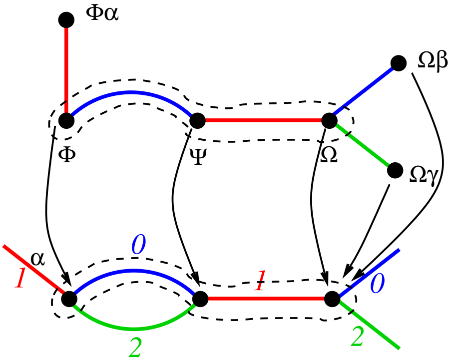

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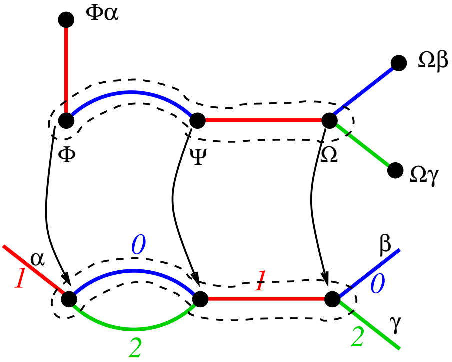

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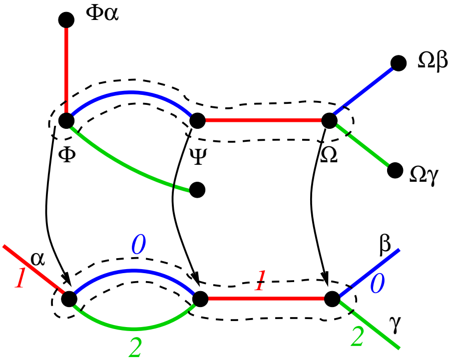

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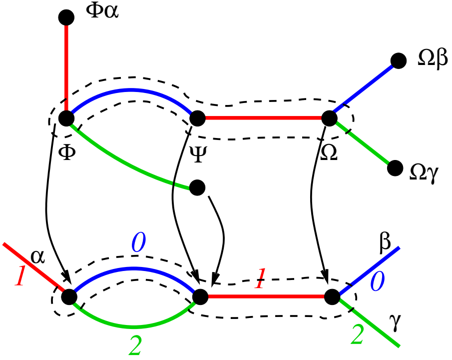

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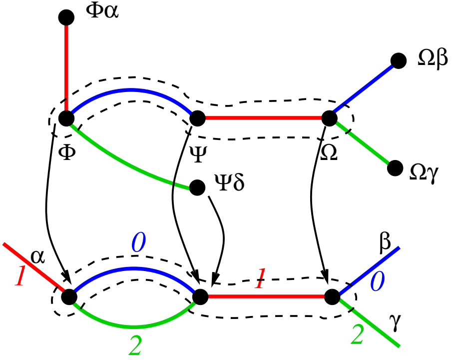

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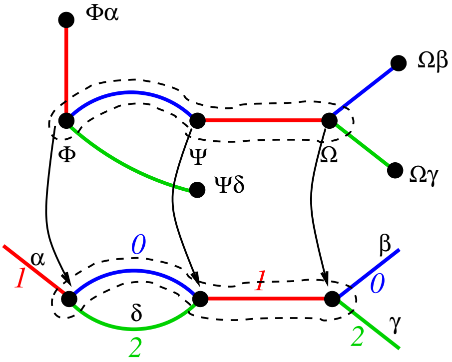

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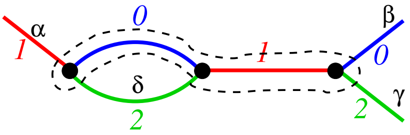

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- *X^ξ is a maniplex iff both previous conditions hold, and whenever $|i - j| > 1$, the voltage of any path of length 4 that alternates colors between i and j is trivial.*

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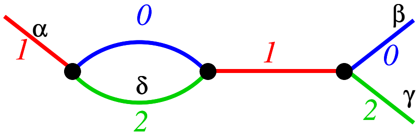
Theorem

Let X be a non-simple maniplex and $\xi : \Pi(X) \to G$ a voltage assigment such that X^ξ is a maniplex. Then X^ξ is the flag graph of a polytope if and only if for every two vertices x and y and any two sets of colors I and J the following condition holds:

$$\xi(\Pi_I^{x,y}) \cap \xi(\Pi_J^{x,y}) = \xi(\Pi_{I\cap J}^{x,y}).$$

Example



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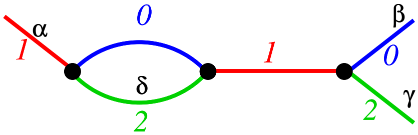

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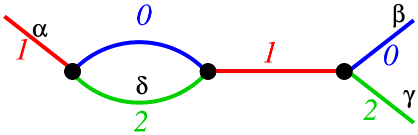

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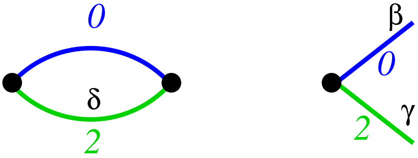
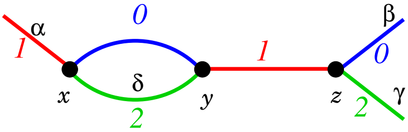
- Simple: $\alpha, \beta, \gamma, \delta \neq 1$ and $\beta \neq \gamma$. In particular α, β and γ are involutions.

Example

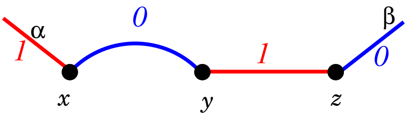


- Connected: $G = \langle \alpha, \beta, \gamma, \delta \rangle$.
- Simple: $\alpha, \beta, \gamma, \delta \neq 1$ and $\beta \neq \gamma$. In particular α, β and γ are involutions.
- Maniplex: $\delta^2 = 1$ and $(\beta\gamma)^2 = 1$.

Example

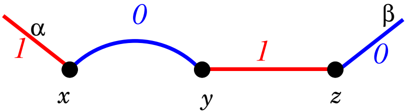


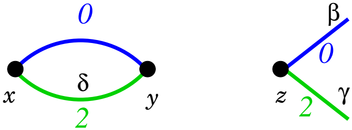
Example



- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.

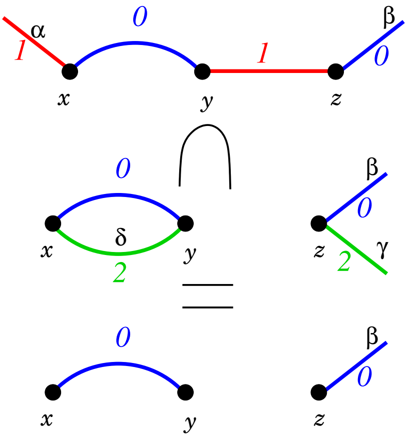
Example





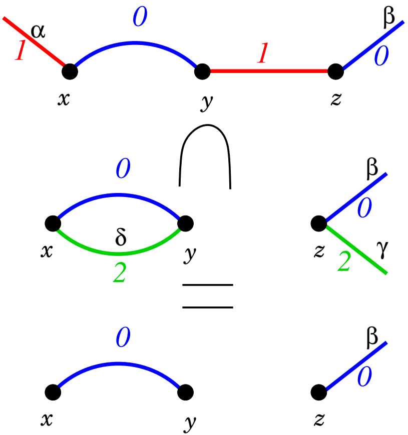
- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x}_{\{0,2\}}) = \langle \delta \rangle$.

Example



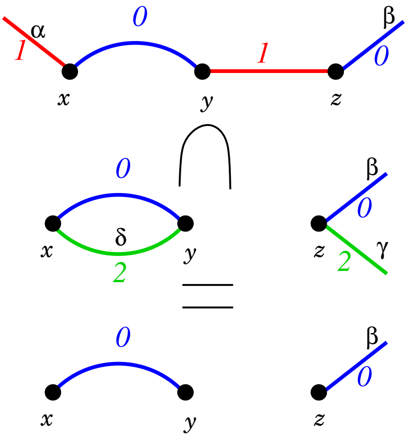
- $\xi(\Pi^x_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^x_{\{0,2\}}) = \langle \delta \rangle$.
- $\xi(\Pi^x_{\{0\}}) = 1$.

Example



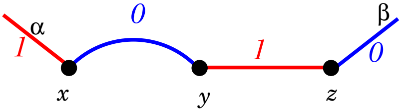
- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x}_{\{0,2\}}) = \langle \delta \rangle$.
- $\xi(\Pi^{x}_{\{0\}}) = 1$.
- $\therefore \langle \alpha, \beta \rangle \cap \langle \delta \rangle = 1$.

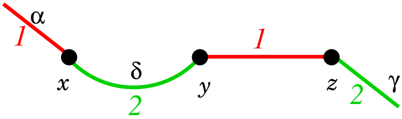
Example



- $\xi(\Pi^{z}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{z}_{\{0,2\}}) = \langle \beta, \gamma \rangle$.
- $\xi(\Pi^{z}_{\{0\}}) = \langle \beta \rangle$.
- $\therefore \langle \alpha, \beta \rangle \cap \langle \beta, \gamma \rangle = \langle \beta \rangle$.

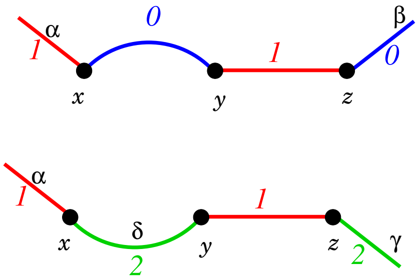
Example





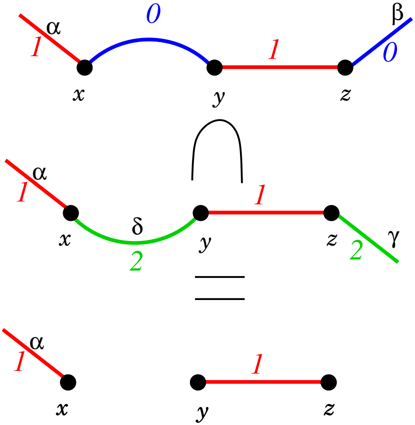
- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.

Example



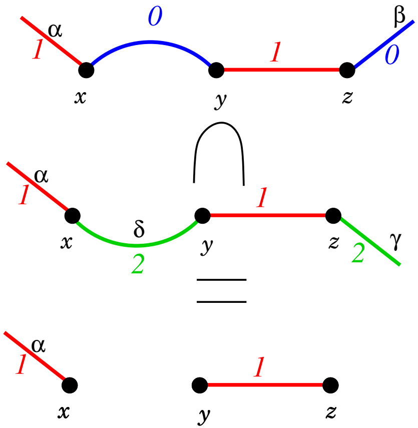
- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x}_{\{1,2\}}) = \langle \alpha, \gamma^{\delta} \rangle$.

Example



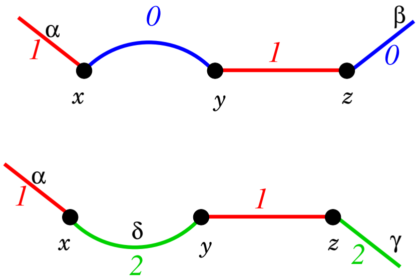
- $\xi(\Pi^x_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^x_{\{1,2\}}) = \langle \alpha, \gamma^\delta \rangle$.
- $\xi(\Pi^x_{\{1\}}) = \langle \alpha \rangle$.

Example



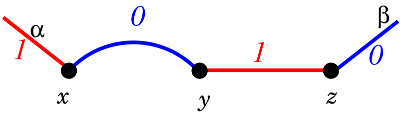
- $\xi(\Pi^{x}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x}_{\{1,2\}}) = \langle \alpha, \gamma^{\delta} \rangle$.
- $\xi(\Pi^{x}_{\{1\}}) = \langle \alpha \rangle$.
- $\therefore \langle \alpha, \beta \rangle \cap \langle \alpha, \gamma^{\delta} \rangle = \langle \alpha \rangle$.

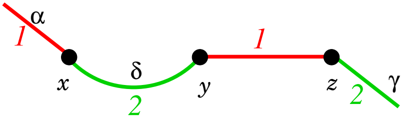
Example



- $\xi(\Pi^{x,y}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.

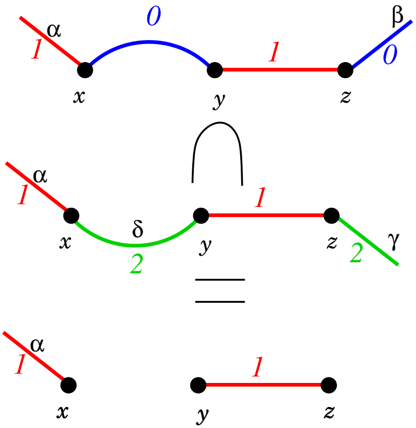
Example





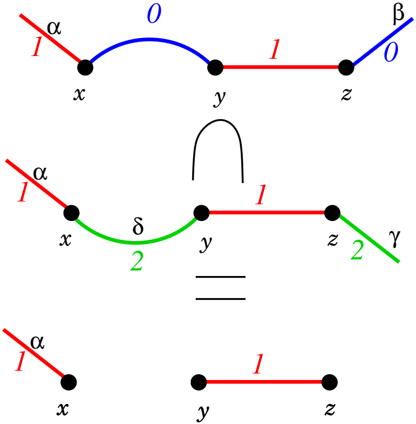
- $\xi(\Pi^{x,y}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x,y}_{\{1,2\}}) = \delta \langle \alpha, \gamma^{\delta} \rangle$.

Example



- $\xi(\Pi^{x,y}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x,y}_{\{1,2\}}) = \delta\langle \alpha, \gamma^{\delta} \rangle$.
- $\xi(\Pi^{x,y}_{\{1\}}) = \emptyset$.

Example



- $\xi(\Pi^{x,y}_{\{0,1\}}) = \langle \alpha, \beta \rangle$.
- $\xi(\Pi^{x,y}_{\{1,2\}}) = \delta\langle \alpha, \gamma^{\delta} \rangle$.
- $\xi(\Pi^{x,y}_{\{1\}}) = \emptyset$.
- $\therefore \langle \alpha, \beta \rangle \cap \delta\langle \alpha, \gamma^{\delta} \rangle = \emptyset$.

Using this method we get these seven conditions (and many other trivial ones):

- $\langle \alpha, \beta \rangle \cap \langle \delta \rangle = 1$
- $\langle \alpha, \beta \rangle \cap \langle \beta, \gamma \rangle = \langle \beta \rangle$
- $\langle \alpha, \beta \rangle \cap \langle \alpha, \gamma^{\delta} \rangle = \langle \alpha \rangle$
- $\langle \alpha, \beta \rangle \cap \langle \alpha^{\delta}, \gamma \rangle = 1$
- $\langle \alpha, \beta \rangle \cap \delta \langle \alpha, \gamma^{\delta} \rangle = \emptyset$
- $\langle \delta \rangle \cap \langle \alpha, \gamma^{\delta} \rangle = 1$
- $\langle \beta, \gamma \rangle \cap \langle \alpha^{\delta}, \gamma \rangle = \langle \gamma \rangle$

Using this method we get these seven conditions (and many other trivial ones):

- $\langle\alpha,\beta\rangle \cap \langle\delta\rangle = 1$
- $\langle\alpha,\beta\rangle \cap \langle\beta,\gamma\rangle = \langle\beta\rangle$
- $\langle\alpha,\beta\rangle \cap \langle\alpha,\gamma^{\delta}\rangle = \langle\alpha\rangle$
- $\langle\alpha,\beta\rangle \cap \langle\alpha^{\delta},\gamma\rangle = 1$
- $\langle\alpha,\beta\rangle \cap \delta\langle\alpha,\gamma^{\delta}\rangle = \emptyset$
- $\langle\delta\rangle \cap \langle\alpha,\gamma^{\delta}\rangle = 1$
- $\langle\beta,\gamma\rangle \cap \langle\alpha^{\delta},\gamma\rangle = \langle\gamma\rangle$

Theorem

Let H be a group. There exists a polytope $\mathcal{P}$ such sthat $H < \Gamma(\mathcal{P})$ and $\mathcal{T}(\mathcal{P},H)$ is the graph from the example iff H is generated by four involutions $\alpha,\beta,\gamma,\delta$ such that $(\beta\gamma)^2 = 1$ and the previous intersection properties hold.

Thank you!