Regular self-dual and self-Petrie-dual maps of arbitrary valency

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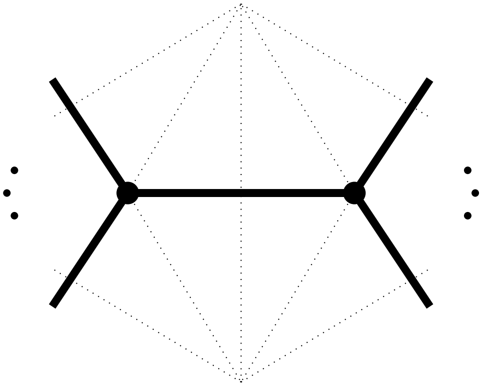
Regular maps - self duality and self-Petrie duality



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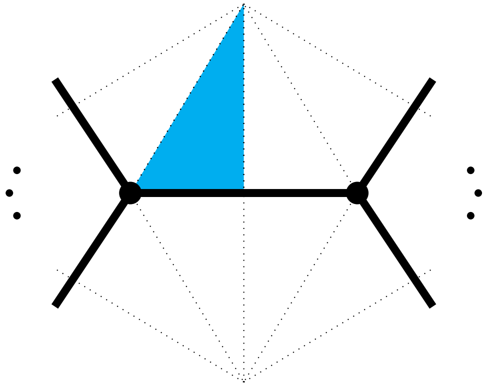

$$G \cong \langle X, Y, Z | X^2, Y^2, Z^2, (XY)^2, (YZ)^k, (ZX)^l, \ldots \rangle$$

where k is the vertex degree, and l is the face length.

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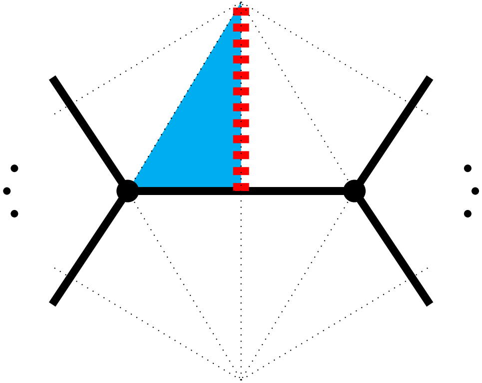

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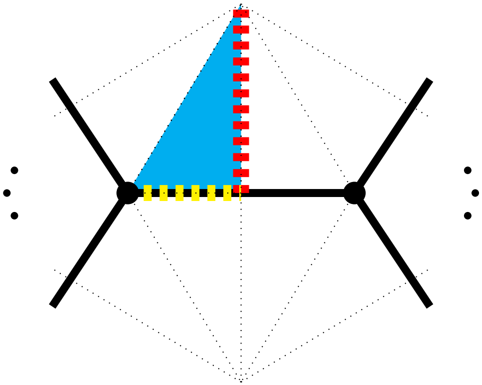

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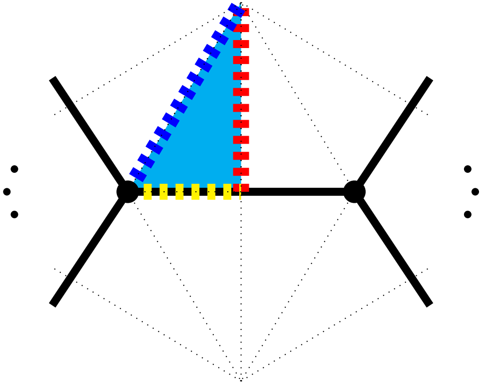

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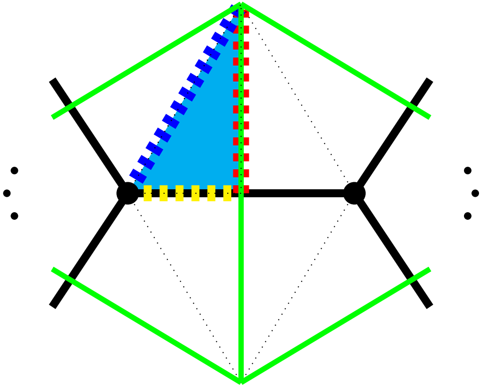

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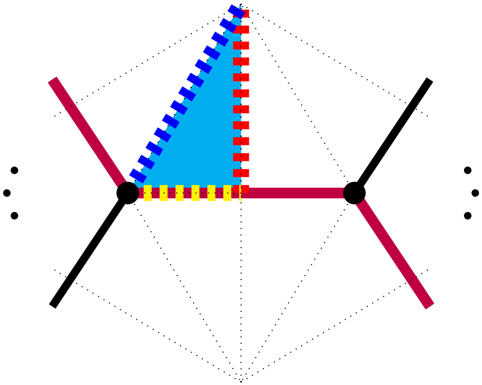

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- Self-Petrie-dual: $\exists$ an automorphism which fixes Y, fixes Z and $X \rightarrow XY$

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Corollary

Let $k \geq 5$ be odd. Suppose that there exists a prime p such that $p \equiv \pm 1 \mod (2k$ and $12)$, and a primitive kth root of unity ζ in a finite field of order p or p^2 such that $3(\zeta + \zeta^{-1}) + 2 = 0$.

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Then there exists a (non-orientable) self-dual and self-Petrie dual regular map of valency k with automorphism group $G \cong PSL(2, p)$.

Algebraic number theory - constructing the finite field

Suppose $k \geq 5$ is prime.

We need to construct a *finite* field in which:
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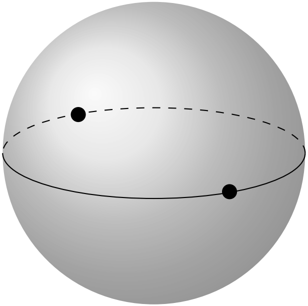
$O(K')/J = F_p$ is our finite field, $\alpha + J$ the element of order k.

Extending the result to multiples of k

Previously - the even valency case...
(Archdeacon, Conder, Širáň 2014)



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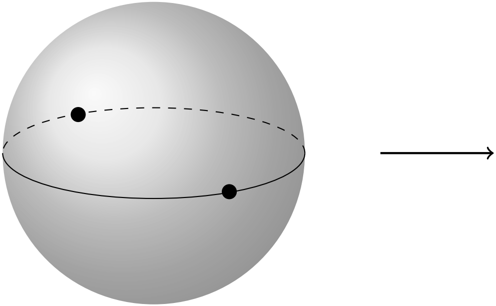

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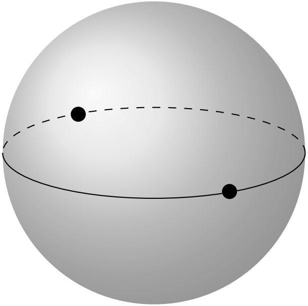

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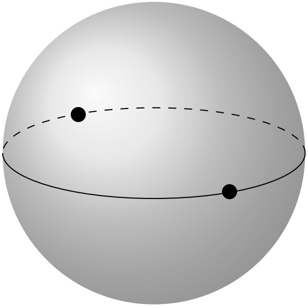

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This method extends analagously to our case:
we want to preserve self-duality and self-Petrie-duality,
the surfaces are non-orientable as the valency is odd.

The missing map...

We need a map for $k = 9$.

No problem!

$G \cong PSL(2, 73)$
$\zeta = 2^4$

Number of vertices 10804.
Genus 27012.

Conclusion

Theorem

For odd $k \geq 5$ there exists a regular self-dual, self-Petrie-dual map with valency k.

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Theorem

For odd $k \geq 5$ there exists a regular self-dual, self-Petrie-dual map with valency k.

Corollary

There is a regular self-dual, self-Petrie-dual map with valency k for all $k \geq 4$.

Table

k	$\mathbf{N}(g)$	**Prime** p
5	-11	11
7	-13	13
9	-73	73
11	263	263
13	-131	131
15	-239	239
17	-4079	4079
19	15503	37 or 419
21	5209	5209
23	-4093	4093
25	56149	56149
27	-16417	16417
29	3161869	59 or 53591

Table: Possible primes p by this method...