Maps Admitting Groups of Automorphisms Acting Regularly on Vertices

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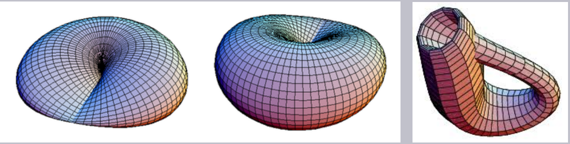
Maps

Definition

A **map** is a 2-cell embedding of a connected graph in a (orientable or non-orientable) surface.



If the surface is orientable, the map is said to be **orientable**.

Interesting maps

Definition

- An orientable map $\mathcal{M}$ is called **orientably regular** if its group of orientation preserving automorphisms acts *regularly* on its set of darts.
- A map is called **regular** if its full automorphism group acts *regularly* on its set of flags.

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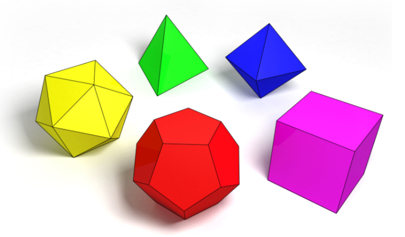

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- if one starts from a **Cayley graph** $C(G, X)$, chooses a **cyclic permutation** p of the generating set X, and defines the local rotation at each vertex by the rule $(g, x) \mapsto (g, p(x))$, then all left-multiplication automorphisms of G lift to map automorphisms and one ends up with an **orientable map that admits an orientation preserving automorphism group acting regularly on the vertices of the map**

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- if, in addition, there exists a **group automorphism** φ of G that preserves X and acts cyclically on X, then choosing $p(x) = \varphi(x)$ gives rise to an **orientably regular map**

Cayley Maps

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Four of the five (orientably regular) Platonic solids are Cayley maps.

Basic observations:

$$|Aut(CM(G,X,p))| \leq |G| \cdot |X|$$

and

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there exists a $\varphi \in Aut(CM(G, X, p))$ such that
$\varphi(1_G) = 1_G$ and $\varphi((1_G, x)) = (1_G, p(x))$

Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

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Theorem (RJ,Širáň)

Let $\mathcal{M} = CM(G, X, p)$ be any Cayley map. Then $\mathcal{M}$ is orientably regular iff there exists a skew-morphism φ of G satisfying the property $\varphi(x) = p(x)$ for all $x \in X$.

Algebraic Properties of Skew-Morphisms

Lemma (RJ, Širáň)

Let φ be a skew-morphism of a group G and let π be the power function of φ. Then the following holds :

1. *the set* $\mathrm{Ker}\varphi = \{g \in G \mid \pi(g) = 1\}$ *is a subgroup of* G*;*
2. $\pi(g) = \pi(h)$ *if and only if* g *and* h *belong to the same right coset of the subgroup* $\mathrm{Ker}\varphi$ *in* G*.*

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Lemma (Conder, RJ, Tucker)

If A is a finite abelian group and φ is a skew-morphism of A, then

1. φ *preserves* $Ker\ \pi$ *setwise;*
2. *the restriction of* φ *to* $Ker\ \pi$ *is a group automorphism.*

Cyclic Extensions from Skew-Morphisms

Let H be a group, and φ be a skew-morphism of H with power function π, and let

$$s(i, b) = \sum_{j=0}^{i-1} \pi(\varphi^j(b)).$$

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Define a multiplication $*$ on $H \times \langle \varphi \rangle$ as follows:

$$(a, \varphi^i) * (b, \varphi^j) = (a\varphi^i(b), \varphi^{s(i,b)+j}),$$

for all $a, b \in H$ and all $i, j \in \mathbb{Z}_{|\varphi|}$.

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Theorem (Conder,RJ,Tucker; Kovács and Nedela)

Let H be a group and φ be a skew-morphism of H of finite order m and power function π. Then the **skew-product** *$A = (H \times \langle \varphi \rangle, *)$ is a group and $H \times \langle \varphi \rangle$ is a complementary factorization of A.*

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for some unique $a' \in A$ and some unique nonnegative integer i less than the order of ρ.

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Parts of this already observed in the 1930's (e.g., Oystein Ore, 1938).

Normal Subgroups of Complementary Cyclic Extensions

Theorem (Conder, RJ, Tucker)

If G is any finite group with a complementary subgroup factorisation $G = AY$ with Y cyclic, then for any generator y of Y, the order of the skew-morphism φ of A is the index in Y of its core in G, or equivalently, the smallest index in Y of a normal subgroup of G.
Moreover, in this case the quotient $\frac{G}{Core_G(Y)}$ is the skew-product group associated with the skew-morphism φ, with complementary subgroup factorisation $A \cdot \frac{Y}{Core_G(Y)}$.

Skew-Morphisms Classifications

Based on the type of the skew-morphism or the base group:

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)
- -1-balanced skew-morphisms on abelian groups that give rise to a regular Cayley map (M. Conder, RJ, T. Tucker)
- t-balanced skew-morphisms of cyclic groups that give rise to a regular Cayley map (Young Soo Kwon)
- t-balanced skew-morphisms on dihedral groups that give rise to a regular Cayley map (Jin Ho Kwak, Young Soo Kwon, and Rongquan Feng)
- t-balanced skew-morphisms on dicyclic groups that give rise to a regular Cayley map (Jin Ho Kwak and Ju-Mok Oh)
- t-balanced skew-morphisms on semi-dihedral groups that give rise to a regular Cayley map (Ju-Mok Oh)
- regular, non-balanced Cayley maps over a dihedral group D_{2n}, n odd (Kovács, Marušič, Muzychuk)
- index 3 skew-morphisms of cyclic groups that give rise to regular Cayley maps (Jun-Yang Zhang)
- coset-preserving automorphisms of cyclic groups (Bachratý, RJ)

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The majority of orbits of skew-morphisms do not generate the whole group and/or are not closed under taking inverses.

Orbits Closed Under Inverses

Lemma (RJ, Nedela)

Let φ be a skew-morphism of G, X be an orbit of φ, and $K = \langle X \rangle$. Then $\varphi|_K$ preserves K setwise and is also a skew-morphism of K.

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Corollary

Skew-morphism orbits X that are **closed under inverses** *give rise to regular Cayley maps on subgroups of G:*

$$Cay(\langle X \rangle, \varphi|_{\langle X \rangle}, \varphi|_X)$$

The Structure of Orbits of Skew-Morphisms

Lemma (RJ, Nedela)

Let φ be a skew-morphism of a finite group G, and π be its associated power function.
The orbit $\mathcal{O}_a$ of any element a in G under the action of φ, $\mathcal{O}_a = \{a, \varphi(a), \varphi^2(a), \ldots\}$, is

- *either closed under inverses, or*
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- orbits closed under inverses are called **self-paired**
- orbits not closed under inverses are called **paired** with their inverse orbit

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- G is transitive on the vertices and has two orbits of equal size on both edges and darts of $\mathcal{M}$;
  in which case G contains automorphisms mapping any dart to its opposing dart
- G acts transitively on both vertices and edges and has two orbits of equal size on the set of darts of $\mathcal{M}$;
  in which case we talk about a **half-arc-transitive action**

Definition

An orientable map $\mathcal{M}$ will be called *half-regular* if there exists $G \leq Aut\ \mathcal{M}$ acting with two orbits on the darts of the map $\mathcal{M}$ and transitively on the vertices of $\mathcal{M}$.

Half-Regular Maps

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Theorem (RJ, Nedela)

Let G be a group, $\mathcal{M} = CM(G, X, P)$ be a Cayley map of even degree and φ be a skew-morphism of G such that the restriction $\varphi|_X = P^2$. Then $\mathcal{M} = CM(G, X, P)$ is a half-regular Cayley map.

Merging of Orbits

If $\mathcal{O}_a$ and $\mathcal{O}_a$ are two orbits of φ of the same size d whose **union $X = \mathcal{O}_a \cup \mathcal{O}_b$ is closed under inverses and generates all of G**. Then,

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- $X = \mathcal{O}_a \cup \mathcal{O}_b$ gives rise to a Cayley graph $C(G, X)$
- The i-th **alternate merging** P_i of $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ is the sequence $(x_1, y_i, x_2, y_{i+1}, x_3, y_{i+2}, \ldots)$

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Corollary (RJ, Nedela)

Let φ be a skew-morphism of a group G and let $\mathcal{O}_a$ and $\mathcal{O}_b$ be two orbits of φ of length d **whose union $X = \mathcal{O}_a \cup \mathcal{O}_b$ is closed under inverses and generates G.**
Then, either both $\mathcal{O}_a$ and $\mathcal{O}_b$ are self-paired, or $\mathcal{O}_a$ and $\mathcal{O}_b$ are paired and $\mathcal{O}_b = \mathcal{O}_{a^{-1}}$.
In either case, the Cayley map $CM(G, X, P_i)$ is half-regular, for any $1 \leq i \leq d$.

Example

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  with skew-morphism $(0)(1,8,4,5,7,2)(3,6)$ whose kernel is the 3-subgroup $\langle 3 \rangle$, and $\pi(1) = \pi(4) = \pi(7) = 5$ while $\pi(8) = \pi(5) = \pi(2) = 3$

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- the second merging $(1,5,4,2,7,8)$ results in a similar regular map
- the third merging $(1,2,4,8,7,5)$ is the balanced regular Cayley map $CM(\mathbb{Z}_9, \{1,2,4,8,7,5\}, (1,2,4,8,7,5))$ whose skew-morphism is the 2-multiplication in $\mathbb{Z}_9$

Characterization of Half-Regular Cayley Maps

Theorem (RJ, Nedela)

Let $\mathcal{M} = CM(G, X, P)$ be a Cayley map. Then $\mathcal{M}$ is half-regular with a half-regular-subgroup H, $G_L \leq H \leq Aut\ \mathcal{M}$, if and only if there exists a skew-morphism φ of G whose restriction to X is equal to P^2.

Corollaries:

Corollary

Let $\mathcal{M} = CM(G, X, P)$ be a half-regular Cayley map with a half-regular subgroup H, $G_L \leq H \leq Aut\ \mathcal{M}$. Then one of the following happens:

1. *The group H acts with two orbits on the edges of $\mathcal{M}$ if and only if the two orbits of P^2 on X are both self-paired.*
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Corollary

A Cayley map $\mathcal{M} = CM(G, X, P)$ is half-regular but not regular if and only if there exists a skew-morphism φ of G such that $\varphi|_X = P^2$ but there is no skew-morphism of G whose restriction to X is equal to P.

Corollaries:

Let χ be the *distribution of inverses* function from X into the set $\{0, 1, 2, \ldots, |X| - 1\}$ that maps every $x \in X$ to the smallest non-negative integer i satisfying the property $P^i(x) = x^{-1}$.

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Corollary

Let $\mathcal{M} = CM(G, X, P)$ be a half-regular Cayley map with a half-regular-subgroup H, $G_L \leq H \leq Aut\ \mathcal{M}$. Then the valency $|X|$ of $\mathcal{M}$ must be even and $\chi(x) \equiv \chi(y) \pmod 2$, for all $x, y \in X$.

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Unlike the case of regular maps, proper half-regular maps do not exist for just any distribution of inverses.

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Corollary

Let $\mathcal{M} = CM(G, X, P)$ be a Cayley map satisfying the property that P^2 is 1-balanced, i.e., $P^2(x^{-1}) = (P^2(x))^{-1}$, for all $x \in X$. Then $\mathcal{M}$ is half-regular with a half-regular-subgroup H, $G_L \leq H \leq Aut\ \mathcal{M}$, if and only if there exists a group automorphism φ of G whose restriction to X is equal to P^2.

Corollaries:

A Cayley map $CM(G, X, P)$ is called *t-balanced* if $P(x^{-1}) = (P^t(x))^{-1}$, for all $x \in X$.

Corollary

Let $\mathcal{M} = CM(G, X, P)$ be a Cayley map satisfying the property that P^2 is 1-balanced, i.e., $P^2(x^{-1}) = (P^2(x))^{-1}$, for all $x \in X$. Then $\mathcal{M}$ is half-regular with a half-regular-subgroup H, $G_L \leq H \leq Aut\ \mathcal{M}$, if and only if there exists a group automorphism φ of G whose restriction to X is equal to P^2.

A perfect analogue of the result of Škoviera and Širáň for balanced regular Cayley maps

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- prove that almost all Cayley maps are chiral (?)

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- if the Cayley group of an orientable generalized Cayley map does not consist of orientation preserving automorphisms only (i.e., it is not a Cayley map), it contains an orientation preserving subgroup of index 2
- the underlying graphs of both orientable and non-orientable generalized Cayley maps are Cayley graphs $C(G, X)$, and the Cayley group always acts on the vertices of the map via left multiplication, G_L

An orientable generalized Cayley map may admit both an orientation preserving Cayley automorphism group and an orientation reversing Cayley automorphism group.

Orientable Generalized Cayley Maps that Are Not Cayley

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Theorem (R.J., Širáň, Wang, 2016+)

All **orientable generalized Cayley maps which are not Cayley** *are of the form* $GCM(G, K, X, p)$*, where* $C(G, X)$ *is a bipartite Cayley graph, i.e.,* G *has a subgroup* K *of index* 2 *and* $X \subseteq G - K$*, all elements in* K *are associated with a fixed local permutation* p*, and all elements in* $G - K$ *are associated with* p^{-1}*.*

Non-Orientable Generalized Cayley Maps

Theorem (Kwak and Kwon, 2006)

All **non-orientable generalized Cayley maps** $GCM(G, X, \kappa, f)$ *are of the form* $\mathcal{M} = (\mathcal{F}, \lambda, \rho, \tau)$*:*

- *the underlying graph is a d-valent Cayley graph $C(G, X)$*
- $\kappa : [d] \to [d]$ *is the inverse distribution function* $x_{\kappa(i)} = x_i^{-1}$
- $f : [d] \to \{-1, 1\}$ *satisfies the condition* $f(i) = f(\kappa(i))$*, for all* $i \in [d]$
- *the flag set* $\mathcal{F} = G \times [d] \times \{-1, 1\}$*,*
- $\lambda(g, i, j) = (gx_i, \kappa(i), -f(i)j)$ *(longitudinal involution),*
- $\rho(g, i, j) = (g, i + j, -j)$ *(rotary involution), and*
- $\tau(g, i, j) = (g, i, -j)$ *(transversal involution).*

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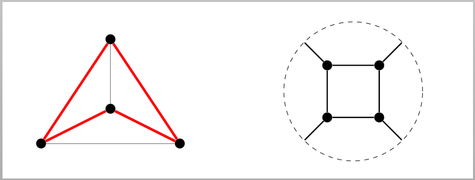

This Slide Dedicated to Steve

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Petrie dual of a generalized Cayley map is a generalized Cayley map

Petrie Dual of an Orientable Generalized Cayley Map

Lemma

If $\mathcal{M}$ is an orientable map, then $P(\mathcal{M})$ is orientable if and only if the underlying graph of M is bipartite.

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Theorem (R.J., Širáň and Wang, 2016+)

1. *The Petrie dual of a Cayley map $CM(G, X, p)$ whose underlying Cayley graph $C(G, X)$ is bipartite with an index 2 subgroup K is the orientable non-Cayley generalized Cayley map $GCM(G, K, X, p)$.*
2. *The Petrie dual of $GCM(G, K, X, p)$ is the Cayley map $CM(G, X, p)$ whose underlying Cayley graph $C(G, X)$ is bipartite.*

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¡Gracias!


26. 8. – 30. 8. 2019
EUROCOMB 2019 in Bratislava
Algebraic graph theory is welcome!

Barcelona, Berlin, Bordeaux, Budapest, Bergen

Bratislava!