CPR graphs of toroidal regular maps of type $\{4, 4\}$

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CPR graphs

Let G be the automorphism group of a regular polytope.

$$G = \langle \rho_0, \ldots, \rho_{d-1} \rangle$$

Suppose that G has degree n.

The *CPR graph* $\mathcal{G}$ of G is a graph with n vertices and with an i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$ (2008, D. Pellicer).

CPR graphs can be used to classify regular polytopes of a given group and for this reason they are an important tool to study polytopes.

Results that were accomplished using CPR graphs

- (2011, Fernandes, Leemans) The symmetric group S_n ($n \geq 4$) is the group of symmetries of a polytope of any rank between 3 and $n-1$;
- (2011, Fernandes, Leemans) There are exactly two polytopes of rank $r = n-2$ for S_n;
- (2018+, Fernandes, Leemans, Mixer) List of all polytopes of rank $r = n-3$ for S_n;
- (2017, Cameron, Fernandes, Leemans, Mixer) The maximal rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$.
- (2018+, Fernandes, Leemans, Weiss) A construction of locally toroidal regular hypertopes combining the CPR graphs of $\{3,6\}_{(2,0)}$ and $\{3,6\}_{(s,0)}$. CPR graphs correspond to faithful permutation representations.

Toroidal regular maps of type $\{4,4\}$

Consider the regular tessellation $\{4,4\}$ of the plane by identical squares. The full symmetry group of it is the Coxeter group $[4,4]$ generated by three reflections ρ_0, ρ_1 and ρ_2.

Identifying opposite sides of the square we obtain a finite toroidal map $\{4,4\}_{(s,t)}$, having $V = s^2 + t^2$ vertices, $2V$ edges and V faces.

ρ_0, ρ_1 and ρ_2 are reflexions of $\{4,4\}_{(s,t)}$ only if $st(s-t) = 0$, that is when case when the map is regular.

There are two families of regular toroidal maps, denoted by $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$.

The group of symmetries of $\{4,4\}$

The group of symmetries of $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$ are factorizations of the Coxeter group $[4,4]$, by

$$(\rho_0\rho_1\rho_2\rho_1)^s = 1 \text{ and } (\rho_0\rho_1\rho_2)^{2s} = 1,$$

of sizes $8s^2$ or $16s^2$, respectively.

For the map $\{4,4\}_{(s,0)}$ consider the unitary translations

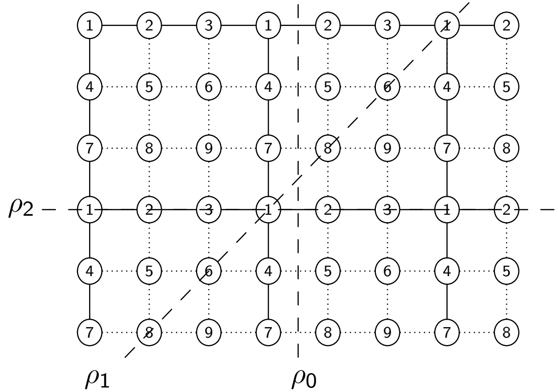
$$g = \rho_0\rho_1\rho_2\rho_1 \text{ and } h = g^{\rho_1}.$$

In the case of the map $\{4,4\}_{(s,s)}$, consider

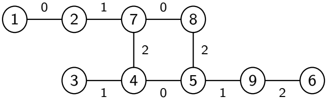
$$g = (\rho_0\rho_1\rho_2)^2 \text{ and } h = g^{\rho_0}.$$

$$U = \langle g, h\rangle$$

$\{4,4\}_{(3,0)}$



1 2 3 1 2 3 1 2
4 5 6 4 5 6 4 5
7 8 9 7 8 9 7 8
ρ_2
1 2 3 1 2 3 1 2
4 5 6 4 5 6 4 5
7 8 9 7 8 9 7 8
ρ_1
ρ_0



1 0 2 1 7 0 8
2 2
3 1 4 0 5 1 9 2 6

CPR graphs on cells, flags or darts

Toroidal maps have faithful actions on the cells (vertices, edges and faces), on the flags and on the darts (edges with a direction).

Faithful permutation representations are in correspondence to core-free subgroups.

Core-free subgroup:	$\{1\}$	$\langle \rho_1 \rangle$	$\langle \rho_0, \rho_2 \rangle$	$\langle \rho_0, \rho_1 \rangle$
$\{4,4\}_{(s,0)}$ $(s \geq 2)$	$8s^2$	$4s^2$	$2s^2$	s^2
$\{4,4\}_{(s,s)}$	$16s^2$	$8s^2$	$4s^2$	$2s^2$

(2005, Li and Širáň) Regular maps that have non-faithful actions on the cells or on darts are identified and they are not regular polyhedra.

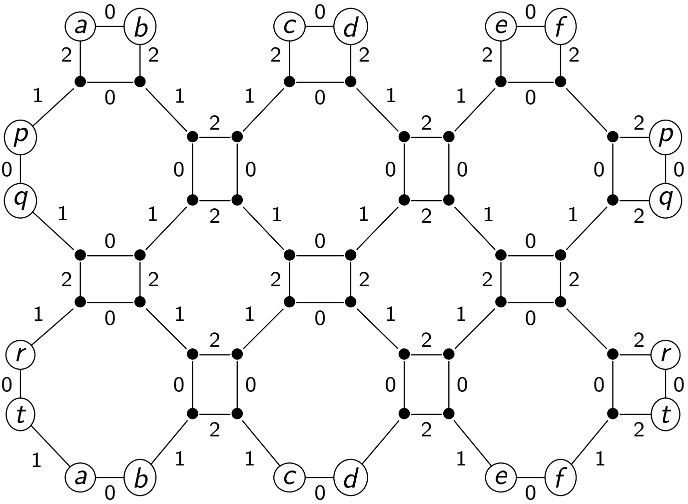
Other core-free subgroups

Let $a,\ b \in \{1, \ldots, s\}$ be such that $s = lcm(a, b)$

	Subgroup	Index
$\{4,4\}_{(s,0)}$	$\langle g^a, h^b \rangle$	$8ab$
	$\langle g^a, h^b \rangle \rtimes \langle \rho_0 \rangle$	$4ab$
	$\langle g^a, h^b \rangle \rtimes \langle \rho_0, \rho_2 \rangle$	$2ab$
$\{4,4\}_{(s,s)}$	$\langle g^a, h^b \rangle$	$16ab$
	$\langle g^a, h^b \rangle \rtimes \langle \rho_1 \rangle$	$8ab$

$$\langle g^a, h^b \rangle \cap \langle g^a, h^b \rangle^{\rho_1} = \langle g^a, h^b \rangle \cap \langle g^b, h^a \rangle = \{1\}$$

CPR graphs of degree $n = 8ab$ for $\{4,4\}_{(s,0)}$



In the example above $a = 3$, $b = 2$ and $s = 6$.

Main result

1. Let $s = lcm(a, b)$.
   - If $s \geq 2$, then n the degree of $\{4,4\}_{(s,0)}$ if and only if

     $$n \in \{s^2, 2ab,\ 4ab,\ 8ab\}.$$

   - n is the degree of $\{4,4\}_{(s,s)}$ if and only if

     $$n \in \{2s^2,\ 4s,\ 4s^2,\ 8ab, 16ab\};$$

2. We describe, up to a conjugacy, all possible core-free subgroups of G;
3. For each possible degree we either give an explicitly CPR graph or an algorithm to obtain it.

Blocks of imprimitivity

1. If U is transitive then $n = s^2$. In this case U is regular and $G \cong U \rtimes G_1$ where G_1 is the stabilizer of the identity.

   For the map $\{4,4\}_{(s,s)}$, U must be intransitive.

2. If $n \neq s^2$ then G is embedded into $S_k \wr S_m$ with
   - $k = ab$ where $s = lcm(a,b)$ and,

     Consider a block B. Let $U_B = \langle g_B \rangle \times \langle h_B \rangle$. The $\langle g_B \rangle$-orbits and the $\langle h_B \rangle$-orbits have all the same size.

   - $m \in \{2,4,8,16\}$ with $m = 16$ only if G is the group of $\{4,4\}_{(s,s)}$.

     We proof that for $\{4,4\}_{(s,s)}$:

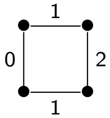
     - if $m = 2$ then $k = s^2$ and

     - if $m = 4$ then $k \in \{s, s^2\}$.

The core-free groups

Consider the map $\{4,4\}_{(s,0)}$ and consider and suppose that U has 4 orbits, then $m = 4$.

- Let $f : G \to S_4$ be the induced homomorphism on the blocks. If $m = 4$ then the $Im(f) \cong D_8$ and the action of $Im(f)$ on the blocks is described by the graph



- Let B a block, up to a conjugacy we may say that $G_B = U \rtimes \langle \rho_0 \rangle$.

  Thus we have the following possibilities for G_x, up to conjugacy and duality,

$$\langle g^a, h^b \rangle \rtimes \langle \rho_0 \rangle, \quad \text{or } \langle g^a, h^b \rangle \rtimes \langle h^{b/2} \rho_0 \rangle$$

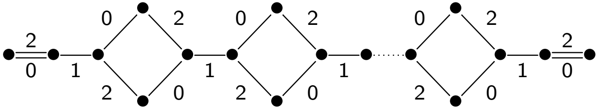
Optimal CPR graphs for $\{4,4\}_{(s,0)}$ and $\{4,4\}_{(s,s)}$

$2s$ is the minimal degree for $\{4,4\}_{(s,0)}$ $(s \geq 3)$.

s odd:

s even:

$4s$ is the minimal degree for $\{4,4\}_{(s,s)}$.



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