Hypertopes with tetrahedral diagram

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Pregeometries (or incidence systems)

Definition (Pregeometry $\Gamma = (X, \mathcal{P}, \mathcal{I})$)

X	set of elements of Γ
$\mathcal{P}$	finite partition of X in types
$\mathcal{I} \subset X \times X$	symmetric relation on X such that $\boxed{\forall X_i \in \mathcal{P} \qquad \mathcal{I} \cap (X_i \times X_i) = \varnothing}$ called incidence relation
$\lvert\mathcal{P}\rvert$	rank of Γ denoted by $rank(\Gamma)$.

Definition ($\Gamma = (X, \mathcal{P}, \mathcal{I})$ pregeometry)

$\mathcal{G}(\Gamma) = (X, E)$, where $\boxed{E := \{\{x, y\} : (x, y) \in \mathcal{I}\}}$ is called the incidence graph of Γ.

Remark

Pregeometries are multipartite graphs.

$\Gamma = (X, \mathcal{P}, \mathcal{I})$, $\Lambda = (Y, \mathcal{Q}, \mathcal{J})$ pregeometries.

Definition

$f : X \to Y$ homomorphism from Γ to Λ if

- Elements of the same type are mapped to elements of the same type: $\boxed{\forall X_i \in \mathcal{P} \quad \exists Y_j \in \mathcal{Q} \qquad f(X_i) \subset Y_j}$
- incident elements are mapped to incident elements:

$$\boxed{(x, y) \in \mathcal{I} \Rightarrow (f(x), f(y)) \in \mathcal{J}}$$

Isomorphism: bijective homomorphism f from Γ to Λ such that f^{-1} is a homomorphism from Λ to Γ.

Automorphism of Γ: isomorphism from Γ to Γ.

Symmetries, correlations and dualities

Remarks

- Any isomorphism f from Γ to Λ induces a bijection
$$\boxed{\mathcal{P} \to \mathcal{Q},\ X_i \mapsto f(X_i)} \Rightarrow rank(\Gamma) = rank(\Lambda).$$
- The set $Aut(\Gamma)$ of automorphisms of Γ is a group containing
$$\boxed{Aut_{\mathcal{P}}(\Gamma) = \{f \in Aut(\Gamma) : \forall X_i \in \mathcal{P} \quad f(X_i) = X_i\}}$$
as a subgroup.

Definition

- Symmetries or type preserving automorphisms: elements of $Aut_{\mathcal{P}}(\Gamma)$.
- Correlations: elements of $Aut(\Gamma) \setminus Aut_{\mathcal{P}}(\Gamma)$.
- Dualities: involutory correlations ($f = f^{-1}$).

Example (J. Tits)

Example (Coset pregeometry $\Gamma = \Gamma(G; \{G_i\})$)

G group, $\{G_1, \ldots, G_r\}$ set of r subgroups of G.

$$X = \bigcup_{i=1}^{r} X_i \quad \text{with} \quad X_i = \{G_i g : g \in G\},\ i = 1, \ldots, r.$$

$\mathcal{P} = \{X_1, \ldots, X_r\}$ and

$$\boxed{(G_i g, G_j h) \in \mathcal{I} \quad \text{iff} \quad i \neq j \text{ and } G_i g \cap G_j h \neq \varnothing}$$

Then $\Gamma = (X, \mathcal{P}, \mathcal{I})$ is a pregeometry of rank r and, for any $a \in G$

$$\varphi_a : X \to X,\ \ G_i g \mapsto G_i g a$$

is a symmetry of Γ.

Flags and chambers of a pregeometry $\Gamma = (X, \mathcal{P}, \mathcal{I})$

Definition

Flag (chain): set of pairwise incident elements (clique)

Rem: Any flag contains at most one element of each type.

Definition

A flag F containing elements of all types is called a chamber.

Rem: Chambers are maximal flags (with respect to inclusion).

Definition

If any maximal flag is a chamber then Γ is called an incidence geometry. An incidence geometry $\Gamma = \Gamma(G; \{G_i\})$ is called a coset geometry.

Thin incidence geometries

Definition

An incidence geometry Γ of rank r is called thin if any flag of cardinality $r-1$ is contained in exactly two chambers (called adjacent chambers).

Rem: Thinness is weaker than the diamond condition.

Remark

Any coset pregeometry Γ has chambers and is a coset geometry if $rank(\Gamma) \leqslant 3$.

We are interested in particular examples of rank 4 thin coset geometries.

The residue of a flag

Definition

Let X_F be the set of elements incident with all the elements in a flag F. Restricting $\mathcal{P}$ and $\mathcal{I}$ to X_F one gets a pregeometry Γ_F, called the residue of F.

Rem: Γ thin incidence geometry of rank r
$\Rightarrow \Gamma_F$ thin incidence geometry of rank $r - |F|$.

Definition (residually connectedness)

A thin incidence geometry is called a hypertope if any residue of rank at least 2 is connected.

Rem.: Γ hypertope $\Rightarrow$ Γ connected (since $\Gamma = \Gamma_\varnothing$)

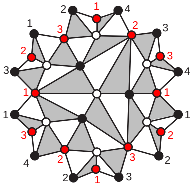
Regular and chiral hypertopes

Theorem

Γ *hypertope* $\Rightarrow Aut_{\mathcal{P}}(\Gamma)$ *acts semi-regularly on chambers.*

Definition

Γ regular if $Aut_{\mathcal{P}}(\Gamma)$ acts regularly on chambers.
Γ chiral if $Aut_{\mathcal{P}}(\Gamma)$ acts with two orbits on chambers and any two adjacent chambers lying in different orbits.



Fano plane embedding of genus 3 (Klein's quartic).

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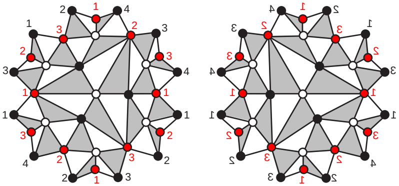

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Distinguished generators

Γ chiral hypertope of rank r, $R = \{1, \ldots, r\}$, $G := Aut_{\mathcal{P}}(\Gamma)$, C chamber of Γ. For $i \in R$ let C^i be the adjacent chamber of C differing from C only in the element of type $X_i \in \mathcal{P}$. Then

- $\forall i, j \in R \quad \exists \alpha_{ij} \in G \quad$ s.t. $\boxed{\alpha_{ij}(C) = (C^i)^j}$
- $\boxed{G = \langle \alpha_{ij} : i, j \in R \rangle}$
- Set $\boxed{\alpha_i := \alpha_{ir}}$. Then $\boxed{G = \langle \alpha_1, \ldots, \alpha_{r-1} \rangle}$ and $\boxed{\alpha_r = 1_G}$ since $\alpha_{ij} = \alpha_{ir}\alpha_{rj} = \alpha_{ir}\alpha_{jr}^{-1} = \alpha_i\alpha_j^{-1}$.
- For $J \subset R$ set $\boxed{G_J := \langle \alpha_i\alpha_j^{-1} : i, j \in J \rangle}$ ($G_\varnothing = \{1_G\}$). Then

$$G_J \cap G_K = G_{J \cap K}$$

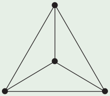
for any $J, K \subset R$ (intersection property: IP^+).

Let $G = \langle S \rangle$ be a group with generating set $S = \{\alpha_1, \ldots, \alpha_{r-1}\}$. We say that (G, S) is a C^+-group if G satisfies IP$^+$ for $R = \{1, \ldots, r\}$ with $\alpha_r = 1_G$.

The complete graph having vertices $\alpha_1, \ldots, \alpha_{r-1}, \alpha_r = 1_G$, where the edge between α_i and α_j is labelled by the order of $\alpha_i \alpha_j^{-1}$ is called the diagram of the C^+-group (G, S).

Convention: We erase edges labelled by 2 and we don't label edges if its label is 3.

Example (Tetrahedral diagram)



$$S = \{\alpha_1, \alpha_2, \alpha_3\}$$
$$|\alpha_1| = |\alpha_2| = |\alpha_3| = 3$$
$$|\alpha_1\alpha_2^{-1}| = |\alpha_2\alpha_3^{-1}| = |\alpha_3\alpha_1^{-1}| = 3$$

Chiral hypertopes from C^+-groups

(G, S) C^+-group, $S = \{\alpha_1, \dots, \alpha_{r-1}\}$, $R = \{1, \dots, r\}$. Set

$$G_i := G_{R\setminus\{i\}} = \begin{cases} \langle \alpha_j : j \neq i \rangle & \text{if } i \neq r \\ \langle \alpha_1^{-1}\alpha_2, \dots, \alpha_1^{-1}\alpha_{r-1} \rangle & \text{if } i = r \end{cases}$$

and set $\Gamma(G, S) := \Gamma(G, \{G_i\})$.

Theorem

*If $\Gamma = \Gamma(G, S)$ is a **hypertope** and $G < Aut_{\mathcal{P}}(\Gamma)$ has **two orbits** on chambers, then Γ is chiral if and only if there is no $\sigma \in Aut(G)$ sending α_i to α_i^{-1} for any $i \in R$. Otherwise, Γ is regular.*

Rem: Hypertopes $\Gamma(G, S)$ with linear diagram are abstract polytopes.

Infinite families with tetrahedral diagram

For $G = PSL(2, q)$ we looked for hypertopes $\Gamma^{(q)} = \Gamma(G, S)$ with tetrahedral diagram $\Rightarrow$ $\boxed{q \equiv 1 \mod 3}$

Results

- For any prime $q \equiv 1 \mod 3$ we give a chiral hypertope.
- For any $q = p^2$ with a prime $p \equiv -1 \mod 3$ we give a regular hypertope.

In both cases:

$$\alpha_1 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \alpha_2 = \pm \begin{pmatrix} e & e^2 \\ 0 & e^2 \end{pmatrix}, \alpha_3 = \pm \begin{pmatrix} e^2 & e \\ 0 & e \end{pmatrix},$$

where e is a primitive third root of unity in $GF(q)$ and

$$G_2, G_3 \cong A_4, \qquad G_1, G_4 \cong E_q : C_3$$

Proper and improper correlations

If $\Gamma^{(q)}$ is chiral (q prime) then there are correlations

- fixing the two orbits of $Aut_{\mathcal{P}}(\Gamma^{(q)}) = PSL(2, q)$ on chambers (proper correlations).
- interchanging the two orbits of $Aut_{\mathcal{P}}(\Gamma^{(q)}) = PSL(2, q)$ on chambers (improper correlations).

Hence $\Gamma^{(q)}$ has simultaneously proper and improper correlations (a property that chiral abstract polytopes can not have).